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LANCHESTER ATTRITION PROCESSES AND THEATER-LEVEL COMBAT MODELS

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
This paper is an expository survey of some aspects of the theory of Lanchester attrition processes and of applications of these processes in models of theater-level combat. Classical Lanchester differentials - including homogeneous and heterogeneous square and linear laws - are discussed. Underlying assumptions and mathematical forms of analogous continuous time, discrete state space Markov attrition processes are presented. A detailed treatment of a class of simplified ("binomial") attrition equations is given, including underlying
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assumptions, exact equations for expected attrition and exponential
approximations to the exact equations. Applications of the binomial and
Markov attrition processes as the basis for attrition calculations in the
CONAF Evaluation Model, the IDAGAM I Model, the Lulejian-I Model and the
VECTOR-2 Model (4 principal documented models of theater-level combat) are
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PREFACE

This paper is an expository survey of some aspects of the theory of Lanchester attrition processes and of applications of these processes in models of theater-level combat. Classical Lanchester differential models--including homogeneous and heterogeneous square and linear laws--are discussed. Underlying assumptions and mathematical forms of analogous continuous time, discrete state space Markov attrition processes are presented. A detailed treatment of a class of simplified ("binomial") attrition equations is given, including underlying assumptions, exact equations for expected attrition and exponential approximations to the exact equations. Applications of the binomial and Markov attrition processes as the basis for attrition calculations in the CONAF Evaluation Model, the IDAGAM I Model, the Lulejian-I Model and the VECTOR-2 Model (4 principal documented models of theater-level combat) are considered.

ACKNOWLEDGMENTS

CONTENTS

0.   INTRODUCTION

The purpose of this paper is to present some aspects of the theory and applications of Lanchester attrition models from two particular and, we believe, complementary viewpoints. On the one hand we will be concerned with identifying and articulating physical and mathematical assumptions that underlie and lead to certain stochastic attrition models, and on the other hand, we will consider use of such models as attrition equations in four principal computerized simulations of ground-air combat at the theater level.

In seeking to provide sets of underlying assumptions for attrition models we hope primarily to facilitate qualitative understanding of individual models, to provide one sensible basis for coherent comparisons of different models, and to promote reasoned application of the models. Traditionally, criteria for understanding and comparing models have included such factors as ability to reproduce historical or exercise data, comparability of results to those of established models, and, of course, military judgment. The latter has become pre-eminent because of a general lack of absolute credibility of all models. Understanding assumptions can, we believe, serve as both an alternative to and a component of military judgment. Since models are used despite the lack of data and despite the lack of credence in their predictive capabilities, it seems essential to take full advantage of any method whereby models may be better understood.

The theater-level combat simulations that are treated in this paper are the CONAF Evaluation Model IV (CEM IV), the IDA Ground-Air Model I (IDAGAM I), the Lulejian-I Theater-Level

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Combat Model, and the VECTOR-2 Theater-Level Combat Model. These have been chosen because all are reasonably well documented and have received substantial use in studies and analyses.

Our presentation in this paper is rather self-contained, except that no results are proved, and the paper is organized in the following manner. Section 1 is an introductory treatment of the classical Lanchester models of combat in terms of systems of differential equations. We have included only the material necessary to understand the later sections of the paper, so this section is not to be regarded as a complete treatment of Lanchester differential equations. Indeed, it is highly selective, but serves the purpose stated above. In Section 2 we discuss stochastic analogues of the differential equation models that are in the form of continuous time Markov processes; the sets of assumptions underlying stochastic analogues of the basic Lanchester models are stated and interpreted. Once again the presentation is selective rather than comprehensive.

Section 3 is, in many ways, the conceptual heart of the paper. In it we describe, in terms of underlying assumptions, exact equations for expected attrition, and approximations to the exact equations, for a number of simplified stochastic models of attrition. The simplifications are that the models are static, with no explicit representation of time, and unilateral. Because of these simplications, these models contain essentially all of the attrition equations used in the four theater-level models named above and in Section 4 we show this is so. The discussion in Section 4 presents, in abstracted but accurate form, the equations used in the four models for attrition calculations and indicates how these equations relate to the simplified models described in Section 3.

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1. CLASSICAL LANCHESTER THEORY

In keeping with the objectives set forth in the Introduction, we restrict our attention to only a few of the many deterministic Lanchester-type differential equations of combat. Excluded are physical effects such as time dependent coefficients and optimal fire allocations; for introductions and further references to these and other effects the reader is referred to [45]. We begin with the homogeneous "square" and "linear" laws proposed by Lanchester himself [31]. As general notation and terminology we establish the following. The two sides are called "Blue" and "Red" (a practice dating at least to [31]). Let $t \geq 0$ denote time and let

$b(t)$ = number of Blue combatants surviving at time t

$r(t)$ = number of Red combatants surviving at time t.

1.1 LANCHESTER SQUARE LAW.

The classical <u>Lanchester square law</u> postulates that combat is described by the differential equations

$$(1.1a) \qquad b'(t) = - k_1 r(t)$$

and

$$(1.1b) \qquad r'(t) = - k_2 b(t) \, ,$$

where k_1 and k_2 are positive constants independent of time. That is, each side's losses are proportional to the currently surviving strength of the opposition. Dimensional analysis indicates that k_1 is in units of Blue combatants per Red combatant per time (with k_2 analogous), so that one should interpret k_1 as the rate at which one surviving Red combatant destroys Blue combatants. Note that (1.1) postulates that this rate depends neither on time nor on the number of Blue combatants available as targets:

$$-\frac{b'(t)}{r(t)} = \text{rate (at time } t \text{) of destruction of}$$
Blue combatants per Red combatant

$$= k_1$$

for all t (unless $b(t) = 0$ or $r(t) = 0$, as discussed below).

Lanchester's rationale for the form of (1.1) is the following [31]:

> *The Conditions of Ancient and Modern Warfare Contrasted.*
> There is an important difference between the methods of
> defence of primitive times and those of the present day
> which may be used to illustrate the point at issue. In
> olden times, when weapon directly answered weapon, the
> act of defence was positive and direct, the blow of sword
> or battleaxe was parried by sword and shield; under modern
> conditions gun answers gun, the defence from rifle-fire is
> rifle-fire, and the defence from artillery, artillery. But
> the defence of modern arms is indirect: tersely, the enemy
> is prevented from killing you by your killing him first, and
> the fighting is essentially collective. As a consequence
> of this difference, the importance of concentration in
> history has been by no means a constant quantity. Under
> the old conditions it was not possible by any strategic
> plan or tactical manoeuvre to bring other than approximately
> equal numbers of men into the actual fighting line; one man
> would ordinarily find himself opposed to one man. Even were
> a general to concentrate twice the number of men on any
> given portion of the field to that of the enemy, the number
> of men actually wielding their weapons at any given instant
> (so long as the fighting line was unbroken), was roughly
> speaking, the same on both sides. Under present-day condi-
> tions all this is changed. With modern long-range weapons--
> fire-arms, in brief--the concentration of superior numbers
> gives an immediate superiority in the active combatant ranks,
> and the numerically inferior force finds itself under a far
> heavier fire, man for man, than it is able to return.

Lanchester goes on to state that

> If [...] we assume equal individual fighting value, and
> the combatants otherwise (as to "cover," etc.) on terms of
> equality, each man will in a given time score, on an average,
> a certain number of hits that are effective; consequently,
> the number of men knocked out per unit time will be directly
> proportional to the numerical strength of the opposing force.

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Various attempts have been made (see [45], e.g.) to derive
(1.1) from more primitive assumptions concerning combat, nor-
mally in terms of "point fire" or "area fire" hypotheses. In
the judgment of the author, all of these attempts are flawed
and are either not rigorous ("area fire" is not a well-defined
mathematical concept) or circular (the "assumptions" represent
only a verbal restatement of (1.1)). The essential difficulty,
it seems, is that combat deals with discrete units, whereas
(1.1) allows fractional attrition. One way to avoid this dif-
ficulty is to introduce stochastic versions of the sort dis-
cussed in Sections 2 and 3 below.

It is possible, nonetheless, to give various interpreta-
tions of (1.1) that are consistent with it but are not viewed
as assumptions leading to it (see also [7] and [45]). Here
are two such interpretations.

POINT FIRE INTERPRETATION. Targets must be destroyed
individually but either targets are sufficiently numerous
or the ability to locate them is sufficiently good that each
attacker (while surviving) locates targets at a constant
rate, so that (1.1) holds.

AREA FIRE INTERPRETATION. Targets may be destroyed
more than one at a time, but are dispersed over some region.
However, that area shrinks as the number of targets decreases
and is always a constant multiple of the number of surviving
targets. Each attacking weapon destroys all targets in a
certain area per unit time.

That the first interpretation is consistent with (1.1) is
clear. For the second, if a is the lethal area produced per Red
combatant per unit of time and if A is the ratio of occupied
area to Blue combatants, then assuming coordination of fire,

$$b'(t) = -\frac{(\text{lethal area produced by Red})(\text{Blue strength at } t)}{(\text{area occupied by Blue at } t)}$$

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$$= -\frac{ar(t)b(t)}{Ab(t)}$$

$$= -\frac{a}{A} r(t) \, ,$$

which is the same as (1.1). Note that both interpretations represent an ability to concentrate attacking forces that is consonant with Lanchester's reasoning.

We stress once more that these are interpretations and not sets of underlying assumptions.

Some mathematical consequences of (1.1) will now be given. Because of its particular structure, the system (1.1) is easy to solve in closed form. The solution is:

(1.2a) $\qquad b(t) = b(0) \cosh \lambda t - \alpha r(0) \sinh \lambda t$

(1.2b) $\qquad r(t) = r(0) \cosh \lambda t - \alpha^{-1} b(0) \sinh \lambda t \, ,$

where $\lambda = (k_1 k_2)^{1/2}$ and $\alpha = (k_1/k_2)^{1/2}$. Both the physical interpretations of k_1 and k_2 and the form of the solutions (1.2) dictate that one should interpret λ as a measure of the overall intensity of the combat and α as the relative per combatant effectiveness (one Red combatant relative to one Blue combatant). The equations (1.2) make sense only for t less than or equal to the termination time τ given by

$$\tau = \inf\{t: b(t) = 0 \text{ or } r(t) = 0\} \, ,$$

when one side or the other is annihilated and the combat ceases. It can be shown that

(1.3) $\qquad \tau = \dfrac{1}{2\lambda} \log \dfrac{k_2^{1/2} b(0) + k_1^{1/2} r(0)}{|k_2^{1/2} b(0) - k_1^{1/2} r(0)|} \, ,$

which we take to be infinite if the denominator is zero.

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From (1.1) it follows easily that

$$(1.4) \qquad k_2 b(t)^2 - k_1 r(t)^2 = k_2 b(0)^2 - k_1 r(0)^2$$

for all t; it is from this relationship (i.e., the fact that $k_2 b^2 - k_1 r^2$ is an invariant of the system (1.1)) that the name "square law" derives. As consequences of (1.3) and (1.4) one has the following observations:

1) If $k_1 r(0)^2 > k_2 b(0)^2$, then $\tau < \infty$, $b(\tau) = 0$ and $r(\tau) = [r(0)^2 - \alpha^{-2} b(0)^2]^{1/2} > 0$ (i.e., Red wins a fight to the finish).

2) If $k_1 r(0)^2 = k_2 b(0)^2$, then $\tau = \infty$ and $b(t) \to 0$, $r(t) \to 0$ as $t \to \infty$ (the combat leads to a drawn state of mutual annihilation).

3) If $k_1 r(0)^2 < k_2 b(0)^2$, then $\tau < \infty$, $b(\tau) = [b(0)^2 - \alpha^2 r(0)^2]^{1/2} > 0$ and $r(\tau) = 0$ (Blue wins).

Some additional mathematical relationships arising from (1.1) will be presented following a discussion of the linear law, for which the corresponding relationships will also be presented.

1.2 LANCHESTER LINEAR LAW. By contrast with (1.1), the classical Lanchester linear law is given by

$$(1.5a) \qquad b'(t) = - c_1 b(t) r(t)$$

and

$$(1.5b) \qquad r'(t) = - c_2 r(t) b(t) ,$$

where c_1 and c_2 are positive constants. The units of c_1 are not the same as those of the constant k_1 appearing in (1.1a);

c_1 is in units of "per Red combatant per unit time," i.e., the numerator is dimensionless. One way to interpret this is to rewrite (1.5a) as

$$\frac{b'(t)}{b(t)} = - c_1 r(t) \; ,$$

which shows that c_1 can be thought of as fractional (or percent) Blue attrition per Red combatant per unit time. From (1.5a) we may also observe that

$$- \frac{b'(t)}{r(t)} = \text{rate of destruction of Blue combatants per Red combatant (at time t)}$$
$$= c_1 b(t) \; ,$$

which is directly proportional to surviving Blue strength, not independent of it as was the case for the square law.

Lanchester envisioned the linear law (1.5) as applicable to combat situations in which concentration of attacking forces (of the kind implicit in the previously given interpretations of the square law) is impossible. In his own words [31]:

> The *Hypothesis Varied.* Apart from its connection with the main subject, the present line of treatment has a certain fascination, and leads to results which, though probably correct, are in some degree unexpected. If we modify the initial hypothesis to harmonise with the conditions of long-range fire, and assume the fire concentrated on a certain area known to be held by the enemy, and take this area to be independent of the numerical value of the forces, then [(1.5) ensues and] the rate of loss is independent of the numbers engaged, and is directly as the efficiency of the weapons. Under these conditions the fighting strength of the forces is directly proportional to their numerical strength; there is no direct value in concentration, *qua* concentration, and the advantage of rapid fire is relatively great. Thus in effect the conditions approximate more closely to those of ancient warfare.

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It is possible to give the following interpretations (not sets of underlying assumptions!) for the linear law.

POINT FIRE INTERPRETATION. Targets must be attacked individually but are either sufficiently few or sufficiently difficult to locate and attack that each attacking weapon engages targets at a rate proportional to the number of targets present, so that (1.5) holds.

AREA FIRE INTERPRETATION. Targets may be destroyed more than one at a time but are dispersed over some region that does not vary over time, although targets redisperse between shots. Each attacking weapon generates lethal area, all targets in which are destroyed, at a constant rate per unit of time.

To justify the latter interpretation, let a be the lethal area produced per Red combatant per unit of time and let D be the (constant!) area of the region over which Blue combatants are dispersed. Then (again assuming coordination of fire)

$$b'(t) = - \frac{\text{(lethal area produced by Red)(Blue strength at t)}}{\text{(area occupied by Blue at t)}}.$$

$$= - \frac{a}{D} r(t)b(t),$$

which is of the form (1.5). Note that both interpretations do represent inability of the attacking side to concentrate its fire.

We proceed to a mathematical discussion of (1.5). Despite the nonlinear form of (1.5), it can be shown that when $c_1 r(0) \neq c_2 b(0)$ the solution is given by

$$(1.6a) \qquad b(t) = \left[b(0)^{-1} e^{\gamma t} + c_2 \gamma^{-1} (e^{\gamma t} - 1) \right]^{-1}$$

and by

$$(1.6b) \qquad r(t) = \left[r(0)^{-1} e^{-\gamma t} - c_1 \gamma^{-1} (e^{-\gamma t} - 1) \right]^{-1},$$

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where $\gamma = c_1 r(0) - c_2 b(0)$ is a measure of the initial discrepancy between the two forces. When $c_1 r(0) = c_2 b(0)$ the solution to (1.5) is given by

$$(1.6c) \qquad\qquad b(t) = \frac{b(0)}{1 + c_2 b(0)t}$$

and by

$$(1.6d) \qquad\qquad r(t) = \frac{r(0)}{1 + c_1 r(0)t} \;.$$

Although these solutions do not seem to be as well known as the solution (1.2) to the square law equations, they are not more complicated.

It may be shown from (1.6), but is more easily shown from (1.5), that

$$(1.7) \qquad\qquad c_2 b(t) - c_1 r(t) = c_2 b(0) - c_1 r(0)$$

for all t, on which expression the name "linear law" is based.

For linear law combats the termination time τ is always infinite, but the following properties hold:

1) If $c_1 r(0) > c_2 b(0)$, then as $t \to \infty$, $b(t) \to 0$ and $r(t) \to r(0) - (c_2/c_1)b(0)$.

2) If $c_1 r(0) = c_2 b(0)$, then $b(t) \to 0$ and $r(t) \to 0$.

3) If $c_1 r(0) < c_2 b(0)$, then $b(t) \to b(0) - (c_1/c_2)r(0)$ and $r(t) \to 0$.

The interpretations are as for the square law.

The two previously given sets of interpretations provide one basis for comparison of the square and linear laws, which may be used to choose one or the other for some specific modeling application. Note that the dual interpretations for each law refute, quite unambiguously, some early assertions that the

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square law is applicable only to point fire and the linear law only to area fire. Another way to understand the differences between the two is by means of differences between the sets of assumptions underlying stochastic analogues of these attrition equations. For details, the reader is referred to Sections 2 and 3 below.

FORCE AND ATTRITION RATIOS. In view of the attention given to force ratios and attrition ratios, we conclude our discussion of the homogeneous square and linear laws with comparisons in terms of attrition ratios and derivatives of force ratios. For the former, we have for the square law

$$(1.8a) \qquad \frac{b'(t)}{r'(t)} = \frac{k_1}{k_2} \frac{r}{b} \, ,$$

while for the linear law,

$$(1.3b) \qquad \frac{b'(t)}{r'(t)} = \frac{c_1}{c_2} \, .$$

These expressions follow immediately from (1.1) and (1.5), respectively. For the square law, the attrition ratio is proportional to the inverse of the force ratio, so that the superior side continues to increase its superiority, which is consistent with Lanchester's arguments concerning concentration of forces. On the other hand, for the linear law, the attrition ratio remains constant over time. This difference might also be used as the basis for choosing one law to use in modeling a particular form of combat.

The situation for force ratios is more complicated, but not intractable. Using (1.1) and (1.4) one can show after some calculation that for the square law

$$(1.9a) \qquad (\frac{b}{r})'(t) = \frac{k_2 b(0)^2 - k_1 r(0)^2}{r(t)^2}$$

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(this is the change in the force ratio, rather than the ratio of the changes in the forces). Similarly, (1.5) and (1.7) imply that for the linear law

$$(1.9b) \qquad (\tfrac{b}{r})'(t) = (c_2 b(0) - c_1 r(0)) \frac{b(t)}{r(t)} \ .$$

In particular, for the linear law the force ratio is an exponential function of time. For both cases, if the forces are initially equally strong, that is, if $k_2 b(0)^2 = k_1 r(0)^2$ in the square law case and if $c_2 b(0) = c_1 r(0)$ in the linear law case, the force ratio remains constant over time.

We conclude this section with a brief discussion of analogues of the square law (1.1) and the linear law (1.5) for combats involving heterogeneous forces. Let

M = number of types of Blue combatants,

N = number of types of Red combatants,

$b_i(t)$ = number of Blue combatants of type i surviving at time t,

$r_j(t)$ = number of Red combatants of type j surviving at time t.

1.3 HETEROGENEOUS SQUARE LAWS. By formal analogy with (1.1), one version of the heterogeneous square law asserts that

$$(1.10a) \qquad b_i'(t) = - \sum_{j=1}^{N} k_1(i,j) r_j(t) \ , \qquad i=1,\ldots,M,$$

and

$$(1.10b) \qquad r_j'(t) = - \sum_{i=1}^{M} k_2(j,i) b_i(t) \ , \qquad j=1,\ldots,N,$$

where k_1 and k_2 are matrices (of dimensions $M \times N$ and $N \times M$, respectively) with nonnegative entries, each of which has at

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least one positive entry. Not all of the entries need be posi-
tive, so that some types of Blue combatants, for example, may
be invulnerable to some types of Red combatants. One can
interpret $k_1(i,j)$ as the rate at which one Red combatant of
type j destroys Blue combatants of type i, which not only is
independent of the numbers of various types of Blue combatants
available as targets, but also must take into account fire
directed at other types of Blue combatants.

One must view (1.10) as based purely on analogy with (1.1),
and as lacking clear justification for the linearity in the num-
bers of attacking weapons. In Section 2 we observe that, using
analogous stochastic attrition processes, a somewhat more rigor-
ous and persuasive justification for the functional form (1.10)
can be provided. In particular, it will be seen that linearity
results from a fundamental assumption of mutual probabilistic
independence of various combatants on each side. Thus, (1.10)
is a possible analogue of (1.1).

However, analysis in Section 2 on the basis of sets of
assumptions underlying analogous stochastic attrition processes
indicates that (1.10) is not the proper analogue to (1.1), but
rather that (2.15) below is.

The solution to (1.10) is difficult to obtain in a useful
form; it has, of course, the matrix representation

$$(1.11) \qquad \begin{bmatrix} b(t) \\ r(t) \end{bmatrix} = e^{tK} \begin{bmatrix} b(0) \\ r(0) \end{bmatrix} \quad ,$$

where K is the matrix

$$(1.12) \qquad K = \begin{array}{c} \\ M \\ N \end{array} \begin{bmatrix} \begin{array}{c} M \quad\quad N \end{array} \\ 0 \;\; \vline \;\; k_1 \\ \hline k_2 \;\; \vline \;\; 0 \end{bmatrix} \quad ,$$

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but while (1.11)-(1.12) may be feasible for use in numerical calculations, they are not useful for qualitative purposes. Moreover, there does not, in general, exist a simple "state law" analogous to (1.4), although it can be shown that if the symmetry condition

$$(1.13) \qquad \sum_{\ell=1}^{N} k_2(\ell,i)k_1(i,j) = \sum_{m=1}^{M} k_1(m,j)k_2(j,i)$$

is satisfied for all i and j, then

$$(1.14) \qquad \sum_{i,j} (k_2(j,i)b_i(t)^2 - k_1(i,j)r_j(t)^2)$$
$$= \sum_{i,j} (k_2(j,i)b_i(0)^2 - k_1(i,j)r_j(0)^2)$$

for all t. The condition (1.13) holds, for example if M = N and if all *entries of both* k_1 and k_2 are equal to some fixed constant. In other cases, reduction to a system of two equations is possible [4].

1.4 HETEROGENEOUS LINEAR LAW. The heterogeneous analogue of the linear law (1.5) is the system

$$(1.15a) \qquad b_i'(t) = - b_i(t) \sum_{j=1}^{N} c_1(i,j)r_j(t) , \qquad i=1,\ldots,M,$$

and

$$(1.15b) \qquad r_j'(t) = - r_j(t) \sum_{i=1}^{M} c_2(j,i)b_i(t) , \qquad j=1,\ldots,N,$$

where c_1 and c_2 are matrices with nonnegative entries (and at least one positive entry). The system (1.15) seems to be unsolvable in closed form and essentially nothing appears to be

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known about the qualitative nature of the solution. Of course,
one can solve it numerically in specific modeling problems.

This concludes our discussion of the classical determin-
istic Lanchester models. As previously mentioned, a number
of additional effects such as mixed laws, optimal control and
time dependent coefficients, have not been treated here. Ref-
erences that the reader may wish to consult include [6], [11],
[12], [14], [36], [40], [41], [42], [43], [44], and [53].

2. CONTINUOUS TIME MARKOV ATTRITION PROCESSES

In this section we present some continuous time, discrete state space Markov processes that are analogous to the deterministic Lanchester attrition models discussed in Section 1. We give some emphasis to sets of underlying mathematical assumptions that lead to the stochastic attrition processes in a rigorous manner. Using these assumptions one can understand not only the stochastic processes but also the corresponding deterministic processes.

GENERAL COMMENTS. For background concerning Markov processes we refer the reader to [8] or [18], or any other standard textbook; our exposition matches that in [8]. Let $X = (X_t)_{t \geq 0}$ be a regular continuous time Markov process with countable state space E. Associated with X are:

1) The transition semigroup (P_t) defined by

$$P_t(x,y) = P_x\{X_t = y\} ,$$

where P_x denotes probability under the condition that $X_0 = x$. The transition semigroup satisfies the Chapman-Kolmogorov equation:

$$P_{t+s}(x,y) = \sum_{z \in E} P_t(x,z) P_s(z,y) .$$

2) The infinitesimal generator (matrix) A defined by

$$(2.1) \qquad A(x,y) = \lim_{h \downarrow 0} \frac{P_h(x,y) - I(x,y)}{h} ,$$

where I denotes the identity matrix.

The interpretation of (2.1) is based on the fact that for $x \neq y$ and $t > 0$

$$(2.2) \qquad P\{X_{t+h} = y \mid X_t = x\} = A(x,y)h + o(h)$$

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as $h \downarrow 0$, so that $A(x,y)$ may be regarded as the "rate" of transition of the Markov process from the state x to the state y. That (2.2) holds is a consequence of the forward equation

(2.3) $$P_t' = P_t A .$$

The backward equation

(2.4) $$P_t' = A P_t$$

also holds, but in the context of attrition processes is less intuitive and less useful computationally.

Study of Markov attrition processes was begun by R.N. Snow [39] and has proceeded fitfully since that time. The more important references are the book of Kimball and Morse [36], the dissertation of Clark [9], a series of papers ([16], [17], [48], [49], [50], [51], [52]) by members of the Defence Operational Analysis Establishment in Great Britain and papers of the author ([19] and [26]). The reader is urged to consult these sources for more detailed information than is given in this section.

We begin by describing the basis on which a stochastic attrition process is deemed to be analogous to a deterministic, differential equation model of combat. First we introduce some notation. Let $E = N^2$ be the set of integer pairs (i,j) with $i \geq 0$ and $j \geq 0$. A Markov process (B_t, R_t) with state space E can be regarded as a model of attrition in combat between a homogeneous Blue side and a homogeneous Red side provided that the paths $t \rightarrow B_t$ and $t \rightarrow R_t$ be (with probability one) nonincreasing. That is, only attrition can occur and combatants are counted in integral units. Here

B_t = number of Blue combatants surviving at time t,

R_t = number of Red combatants surviving at time t;

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both of these are random variables for each t. By virtue of the interpretation (2.2) of the infinitesimal generator A of the process (B_t, R_t), the process is called analogous to a differential model of combat *if the form of the generator A and form of the differential equations are sufficiently similar.*

For example, the homogeneous square law process described below has generator A given by

$$(2.5a) \qquad A((i,j),(i-1,j)) = k_1 j$$

and

$$(2.5b) \qquad A((i,j),(i,j-1)) = k_2 i \ ,$$

where k_1 and k_2 are positive constants. For all other states (ℓ,m) except (i,j) we have

$$(2.5c) \qquad A((i,j),(\ell,m)) = 0$$

which, since row sums of a generator matrix are zero, forces

$$(2.5d) \qquad A((i,j),(i,j)) = - (k_2 i + k_1 j) \ .$$

That is, when the attrition process (B_t, R_t) is in state (i,j) (with i Blue survivors and j Red survivors) it can move only to the state $(i-1,j)$, which corresponds to destruction of one Blue combatant, or to the state $(i,j-1)$, corresponding to destruction of one Red combatant. These transitions, by (2.5), occur at the rates $k_1 j$ and $k_2 i$, respectively, and then, using (2.2), the analogy of (2.5a-b) to (1.1a-b) is evident and compelling.

The author's research described in [19] was directed at understanding stochastic attrition processes and facilitating their use in combat simulation models by describing sets of probabilistic assumptions that lead by rigorous mathematical reasoning to stochastic attrition processes with particular

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infinitesimal generators. By comparing assumptions one can understand differences among various processes and more rationally choose a process to use in a given modeling situation. To illustrate this procedure, we now list assumptions for homogeneous square law and linear law processes.

2.1 HOMOGENEOUS SQUARE LAW PROCESS. This is the process (B_t, R_t) with generator A given by (2.5) and ensues from the following assumptions.

1) All combatants on each side are identical.

2) Times between detections by a surviving Red combatant are independent and identically exponentially distributed with mean r_1^{-1}, regardless of the number of surviving Blue combatants (provided that the latter be nonzero).

3) When a Red combatant detects a Blue combatant an instantaneous attack occurs, in which the Blue combatant is destroyed with probability p_1 and survives with probability $1-p_1$. Total loss of contact then takes place immediately.

4) Blue combatants satisfy assumptions 2) and 3) with respective parameters r_2 and p_2.

5. Conditioned on survival, detection and attack processes of all combatants (on both sides) are mutually independent in the probabilistic sense.

To obtain (2.5) one then takes

$$k_i = r_i p_i , \qquad\qquad i=1,2,$$

which can be interpreted (consistently with (1.1), incidentally) as an instantaneous rate of kill.

For a completely rigorous derivation of (2.5) from these assumptions and for further probabilistic properties of the process (B_t, R_t), we refer the reader to [19].

2.2 HOMOGENEOUS LINEAR LAW PROCESS.

This is the Markov attrition process (B_t, R_t) with generator A given by

(2.6a) $\qquad\qquad A((i,j),(i-1,j)) = c_1 ij$,

(2.6b) $\qquad\qquad A((i,j),(i,j-1)) = c_2 ij$,

(2.6c) $\qquad\qquad A((i,j),(i,j)) = -(c_1+c_2)ij$,

and otherwise

(2.6d) $\qquad\qquad A((i,j),(\ell,m)) = 0$.

The analogy to (1.5) is clear. The process results from the following set of assumptions.

1) All combatants on each side are identical.

2) The time required for a particular Red combatant to detect a particular Blue combatant (given survival of both) is exponentially distributed with mean $\tilde{r}_1^{-1}$. Each particular Red combatant detects different Blue combatants independently of one another.

3) A Red combatant attacks every Blue combatant it detects. The engagement is instantaneous, results in destruction of the Blue combatant with probability p_1 and survival of the Blue combatant with probability $1 - p_1$, and is followed by immediate and complete loss of contact. The outcome of the engagement is stochastically independent of all other aspects of the attrition process. Finally, a Red combatant is never harmed in an engagement that it initiates.

4) Blue combatants satisfy assumptions 2) and 3) with respective parameters $\tilde{r}_2$ and p_2.

5) The detection and attack processes of all combatants initially present are mutually independent.

Then, as shown in [19], the equation (2.6) follows with $c_i = \tilde{r}_i p_i$ (i=1,2), the interpretation of which is an instantaneous one-on-one rate of destruction.

As shown in more detail in [19], the linear law process is fundamentally easier to deal with probabilistically because the embedded Markov chain [8] is spatially homogeneous; cf. [19] and [27] for consequences of this observation.

Incidentally, the expressions (2.5) and (2.6) are valid only if $i > 0$ and $j > 0$. If $i = 0$ or $j = 0$, the combat has terminated; mathematically this is represented by making (i,j) an absorbing state with

$$A((i,j),(k,\ell)) = 0$$

for all states (k,ℓ).

By comparing the two sets of assumptions one sees immediately that the fundamental difference between the two processes is that in the square law process the distribution of the time required to detect at least one opponent does not depend on the number of opponents present, whereas in the linear law process the mean time to detect at least one opponent is proportional to the inverse of the number of opponents (i.e., the detection rate, in the sense of the parameter ($=1$/mean) of an exponential distribution, is proportional to the number of opponents). This distinction is in some sense that made by Lanchester, but is now stated in a manner that is more meaningful both physically and mathematically.

Given these sets of assumptions one can use them to decide which process better represents a specific physical combat situation, namely, the one whose assumptions better match the physical reality.

ANALYSIS OF CONTINUOUS TIME MODELS. Once a stochastic attrition model is defined there are various questions one

might pose concerning it. Examples are:

1) For each t, what are the distribution and expectation of B_t and R_t?

2) Do these expectations satisfy the analogous system of differential equations?

3) What are the variances of B_t and R_t and how large are the fluctuations about the expectations?

4) If we let

$$(2.7) \qquad T = \inf\{t: B_t = 0 \text{ or } R_t = 0\} \quad ,$$

which is the termination time in a battle fought to annihilation and is analogous to the time τ in (1.3), what are the distribution and expectation of the terminal state $(B_T, R_T) = (B_\infty, R_\infty)$?

Among the approaches used to answer these questions the following are most prominent.

1) The probabilistic/analytical approach, which attempts to obtain direct answers ([9], [13], [19], [26], [47]).

2) The approach of numerical integration of the forward equation (2.3) in order to obtain the transition semigroup (P_t) ([16], [17], [48], [49], [50], [51], [52]); see also [27]).

3) Use of Monte Carlo simulations ([9], [21], [22]).

4) Derivation of approximations valid for large numbers of combatants ([13], [23], [47]).

5) Use of diffusion approximations ([37], [38]).

Of these, we will illustrate only the first approach and only one aspect of it. The reader is referred to the listed sources for further details. Our illustration is based on the following very general, but also very useful, result.

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THEOREM. (Karr, [27]). Let (B_t, R_t) be a Markov attrition process with generator A, let T be the termination time given by (2.7) and let f be a real-valued function defined on the state space of the process. Then

(2.8)
$$\frac{d}{dt} E[f(B_t, R_t)] = E[Af(B_t, R_t); \{T>t\}] \ .$$

Here the E denotes expectation and the notation $E[X;G]$ means $E[X']$, where X' and X agree on the event G and X' is zero elsewhere.

The expression (2.8) is very general and gives a large number of specific and interesting consequences, a few of which we will list and discuss.

If (B_t, R_t) is the square law process with generator A given by (2.5) and if we take $f(i,j) = i$, then (2.8) gives

(2.9)
$$\frac{d}{dt} E[B_t] = - k_1 E[R_t; \{T>t\}] \ ,$$

an expression first derived by Snow [39]. The equation (2.9) shows that expectations in the square law case do not satisfy the deterministic square law (1.1), but do so approximately for those t small enough that one can neglect the difference

$$E[R_t] - E[R_t; \{T>t\}] = E[R_t; \{T \leq t\}] \ .$$

By successively choosing $f(i,j) = i^2$ and $f(i,j) = j^2$ and performing some algebraic manipulations one obtains from (2.8) the expression

(2.10)
$$\frac{d}{dt} E[k_2(B_t^2 + B_t) - k_1(R_t^2 + R_t)] = 0 \ ,$$

which is the stochastic analogue of the deterministic state equation (1.4).

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Similar, but nicer, results obtain for the linear law process (B_t, R_t) with generator A given by (2.6). In particular,

$$(2.11) \qquad \frac{d}{dt} E[B_t] = - c_1 E[B_t R_t; \{T > t\}]$$

and therefore (by a symmetry argument)

$$(2.12) \qquad \frac{d}{dt} E[c_2 B_t - c_1 R_t] = 0 .$$

Pleasantly, (2.12) is an exact analogue of the linear law state equation (1.7).

Additional results of the sort represented by (2.9)-(2.12) may be found in [27]; see also [21], [22].

A further benefit of identifying sets of underlying assumptions is that generalization to heterogeneous processes can be done easily and plausibly by extending the assumptions, rather than purely formally by writing down similar looking equations. As illustrations, we list the assumptions and generators for heterogeneous square law and linear law processes. Let

M = number of types of Blue combatants,
N = number of types of Red combatants,
$B_i(t)$ = number of Blue combatants of type i surviving at time t,
$B(t) = (B_1(t), \ldots, B_M(t))$,
$R_j(t)$ = number of Red combatants of type j surviving at time t,
$R(t) = (R_1(t), \ldots, R_N(t))$.

Let $E = \mathbf{N}^{M+N}$ be the state space of the process, elements of which will be written as

$$(\ell, m) = (\ell_1, \ldots, \ell_M, m_1, \ldots, m_N) .$$

2.3 HETEROGENEOUS SQUARE LAW PROCESS (TRADITIONAL VERSION).

This is the process $(B(t), R(t))$ with generator A given by

(2.13a) $A((\ell,m),(\ell_1,\ldots,\ell_i-1,\ldots,\ell_M,m_1,\ldots,m_N)) = \sum_{j=1}^{N} k_1(i,j)m_j$

for $i = 1, \ldots, M$; also

(2.13b) $A((\ell,m),(\ell_1,\ldots,\ell_M,m_1,\ldots,m_j-1,\ldots,m_N)) = \sum_{i=1}^{M} k_2(j,i)\ell_i$

for $j = 1, \ldots, N$. The term (2.13a) corresponds to the transition caused by destruction of one Blue combatant of type i; (2.13b) represents destruction of one Red combatant of type j. No other transitions from the state (ℓ,m) are possible.

The analogy of (2.13) to (1.10) is quite clear, but what is particularly persuasive is the analogy of the following set of assumptions to the set leading to the homogeneous square law process. In particular, linearity of the terms (2.13) in the numbers of opposing weapons is a consequence of the independence Assumption 6), which shows that this process cannot represent synergistic effects among combatants on the same side. Here are the assumptions.

1) All combatants of a given type on a given side are identical.

2) Consider a Red combatant of type j. The times between detections it makes of Blue combatants of type i are independent and identically exponentially distributed with mean $r_1(i,j)^{-1}$.

3) For each Red combatant these M ongoing detection processes occur simultaneously and independently.

4) Whenever a Red combatant of type j detects a Blue combatant of type i there occurs an instantaneous attack in which the Blue combatant is destroyed with probability $p_1(i,j)$. Loss of contact follows immediately.

5) Blue combatants satisfy Assumptions 2)-4) with parameters $r_2(j,i)$, $p_2(j,i)$.

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6) Detection and attack processes of all combatants are mutually independent.

It is shown in [19] that these assumptions imply (2.13) with

$$k_1(i,j) = r_1(i,j)p_1(i,j)$$

and

$$k_2(j,i) = r_2(j,i)p_2(j,i).$$

Only a cursory examination of the assumptions reveals that they are unnatural. Assumptions 2) and 3) imply simultaneously occurring detection processes for each type of opposition combatant, independent of the number of opposition combatants of each type, which seems very unreasonable physically in most combat situations. If the total number of detections does not depend on the total number of combatants on the opposing side, then much more plausible would be assumptions to the effect that times between detections (without reference to the type of opponent detected) are merely exponentially distributed with mean $r_1(j)^{-1}$ for type j Red combatants and that when a detection occurs, and if there are ℓ_i currently surviving Blue combatants of type i, then the detected Blue combatant is of type i with probability $\ell_i / \sum_{n=1}^{M} \ell_n$. Making this modification yields the following.

2.4 HETEROGENEOUS SQUARE LAW PROCESS WITH FIRE ALLOCATION.
Assumptions 1), 4) and 6) hold as above, together with the following:

2') Times between detections by a Red combatant of type j are independent and identically exponentially distributed with mean $r_1(j)^{-1}$, regardless of the numbers of Blue combatants present.

3') If a detection occurs when the surviving Blue force has composition $(\ell_1,\ldots,\ell_M)$ the detected combatant is of type i with probability $\ell_i / \sum_{n=1}^{M} \ell_n$.

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5') Blue combatants satisfy assumptions 2'), 3') and 4) with parameters $r_2(i)$ and $\dot{p}_2(j,i)$.

Subject to these assumptions the attrition process $(B(t),R(t))$ has generator A given by

(2.14a) $\quad A((\ell,m),(\ell_1,\ldots,\ell_i-1,\ldots,\ell_M,m_1,\ldots,m_N))$

$$= \frac{\ell_i}{\hat{\ell}} \sum_{j=1}^{N} r_1(j)p_1(i,j)m_j$$

for $i = 1, \ldots, M$ and by

(2.14b) $\quad A((\ell,m),(\ell_1,\ldots,\ell_M,m_1,\ldots,m_j-1,\ldots,m_N))$

$$= \frac{m_j}{\hat{m}} \sum_{i=1}^{M} r_2(i)p_2(j,i)\ell_i \, ,$$

for $j = 1, \ldots, N$, where $\hat{\ell} = \sum \ell_i$ and $\hat{m} = \sum m_j$.

By virtue of the assumptions (which are clearly square law assumptions) the factors $\ell_i/\hat{\ell}$ and $m_j/\hat{m}$ in (2.14) do not make this a linear law process. These factors simply represent a proportional allocation of fire.

The plausibility of the assumptions of this process suggests that (1.10) is not the appropriate deterministic square law in the heterogeneous case, but that instead the appropriate system of equations is

(2.15a) $\quad b_i'(t) = - \dfrac{b_i(t)}{\hat{b}(t)} \sum_{j=1}^{N} k_1^*(i,j)r_j(t), \qquad i=1,\ldots,M$

and

(2.15b) $\quad r_j'(t) = - \dfrac{r_j(t)}{\hat{r}(t)} \sum_{i=1}^{M} k_2^*(j,i)b_i(t), \qquad j=1,\ldots,N \, ,$

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where k_1^*, k_2^* are matrices with positive entries and where $\hat{b}(t) = \sum b_i(t)$, $\hat{r}(t) = \sum r_j(t)$. That (2.15) seems to be more appropriate than (1.10) would have been difficult to discern without identification of sets of underlying assumptions.

One can replace the proportional fire allocation of Assumption 3') by a more general allocation of the form

$$w_{ji}\ell_i \Big/ \sum_{n=1}^{M} w_{jn}\ell_n,$$

where the w_{jn} are weighting factors. Indeed, it is shown in [19] that all allocations satisfying some simple and intuitive hypotheses are of this form. For an illustration we refer to the discussion of the IDAGAM I model in Section 4 below.

We next consider a corresponding linear law model.

2.5 *HETEROGENEOUS LINEAR LAW PROCESS.* This process has generator A given by

(2.16a) $A((\ell,m),(\ell_1,\ldots,\ell_i-1,\ldots,\ell_M,m_1,\ldots,m_N))$

$$= \ell_i \sum_{j=1}^{N} c_1(i,j)m_j$$

for $i = 1, \ldots, M$, and by

(2.16b) $A((\ell,m),(\ell_1,\ldots,\ell_M,m_1,\ldots,m_j-1,\ldots,m_N))$

$$= m_j \sum_{i=1}^{M} c_2(j,i)\ell_i$$

for $j = 1, \ldots, N$, with all other transitions impossible. While the resemblance to (1.15) is clear, the analogy of the following assumptions to the assumptions engendering the homogeneous linear law process is even more compelling.

1) All combatants of a given type on the same side are identical.

2) The time required for a Red combatant of type j to detect a given Blue combatant of type i is exponentially distributed with mean $\tilde{r}_1(i,j)^{-1}$. Each Red combatant detects different Blue combatants independently of one another.

3) A Red combatant attacks every Blue combatant it detects. The attack is instantaneous and destroys a Blue combatant of type i with probability $p_1(i,j)$; immediately thereafter contact is lost.

4) Blue combatants satisfy Assumptions 2) and 3) with parameters $\tilde{r}_2(j,i)$ and $p_2(j,i)$.

5) Detection and attack processes of <u>all</u> combatants are mutually independent.

As proved in [19], (2.16) then follows with $c_1(i,j) = \tilde{r}_1(i,j)p_1(i,j)$ and $c_2(j,i) = \tilde{r}_2(j,i)p_2(j,i)$.

In this case the assumptions àre quite compatible with those for the homogeneous case and no alternate version is necessary.

The continuous time stochastic attrition processes are sufficiently intractable to make their application in iterative simulations of large scale combat discouragingly difficult. One alternative is to choose a small time interval and to approximate one of the deterministic equations from Section 1. Another alternative is to seek to develop equations that are simpler (in the sense, for example, that they contain no explicit representation of time) but are still based on probabilistic assumptions that are felt to be plausible. Such assumptions might be regarded as retaining the essential features of the sets of assumptions presented in this section; alternatively, they might be viewed as more plausible <u>per se</u> than the assumptions underlying the continuous time processes

given above. In the next section we develop a number of simplified stochastic models of attrition.

3. SIMPLIFIED STOCHASTIC ATTRITION MODELS

In this section we present a number of simplified, stochastic attrition models that are related--in terms of the sets of underlying assumptions--to the continuous-time Markov attrition models discussed in the preceding section. The essential simplification in the models of this section (as compared to those of Section 2) is that there is no explicit representation of time. Rather, these are static attrition models that represent the attrition occurring over an interval of time only by the total attrition at the end of that interval. Furthermore, the models presented here are one-sided, describing the attrition to a set of targets inflicted by invulnerable searchers. One way to use such models to describe attrition in bilateral combat is to first view the Blue side as targets and the Red side as searchers (which allows calculation of attrition to the Blue side) and then--using the same initial numbers--to view the Blue side as searchers and the Red side as targets. (Other ways include allowing only the survivors on one side to return fire against the other, or using weighted averages of several different methods.) Since essentially all theater-level combat models are fixed time step, iterative simulations, the models described in this section are well-suited to use in large scale combat simulations and, indeed, are frequently so used, as we will describe in Section 4 below. In such applications, of course, the numerical values of the parameters of the equations must be chosen consistently with the time step of the simulation.

We will now describe several simplified attrition models, each of which is related to one of the stochastic attrition processes discussed in Section 2 and related in two ways to one of the deterministic Lanchester equations given in Section 1. The first relation is through the stochastic process and the second is a direct relation obtained by approximating the functional form of the model by some simpler form. As general

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notation and terminology, we introduce:

s = number of searchers

t = number of targets

Δt = attrition to targets.

In our discussion, s and t are not random; only the attrition Δt is a random variable.

The models we are about to describe have been developed mainly in [20], [24] and [29]; see also [1] and [15]. Several of the approximations noted below, however, have been in widespread use before underlying assumptions were clarified. In the discussions of approximations, when a quantity is termed "small" we mean roughly that its square is sufficiently close to zero that it can be neglected.

3.1 HOMOGENEOUS SQUARE LAW PROCESS.

The assumptions underlying this process are the following.

1) All targets are identical; all searchers are identical.

2) Each searcher detects at least one target with probability d ε (0,1], regardless of the number of targets. For a given searcher each target is equally likely to be in the class of detected targets.

3) A searcher making no detection makes no attack.

4) If a searcher makes at least one detection, it chooses randomly from the targets it has detected, exactly one target to attack. Each detected target is equally likely to be chosen.

5) The probability that a searcher destroys a target, given an attack, is k ε (0,1].

6) The searchers are mutually independent in the probabilistic sense.

The reader will note the essential similarity of the assumptions here to those underlying the homogeneous square law

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process of Section 2: in the latter the distribution of the time required to make a detection is independent of the number of targets, and in the case at hand, the probability of a detection is independent of the number of targets.

From these assumptions it follows that

$$(3.1) \qquad E[\Delta t] = t[1 - (1 - \frac{kd}{t})^s] \; ;$$

the derivation, which is not difficult, is given in [24].

While the analogy to square law processes is quite clear in the context of underlying assumptions, some mathematical approximations to (3.1) help to make the analogy still stronger. When the quantity kd/t is small the approximation

$$(3.2) \qquad E[\Delta t] \simeq t[1 - \exp(- \frac{kds}{t})]$$

is valid. If the quantity kds/t is small then (3.1) may be approximated as

$$(3.3) \qquad E[\Delta t] \simeq kds \; ,$$

which is formally the same as the classical square law (1.1).

We observe that kd/t is small if the number of targets is large (which is consistent with the underlying assumptions; if there are many targets the probability of detecting at least one is not a function of their exact number) or if the detection probability d is sufficiently small. The latter point is important because d is related to the length of the time period over which attrition is being calculated. For short time periods, d can be very small and in this case (3.2) is a useful approximation. It will be seen throughout this section that approximations--especially exponential approximations--all improve as the appropriate detection probabilities decrease. Of course, if the probability k of target destruction given attack is

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small, this might also render (3.2) valid, but k is essentially independent of the time period under consideration and is not necessarily small in realistic situations.

Finally, we point out that validity of (3.2) is independent of the value of s, so that (3.2) holds even for large values of s, whereas (3.3) cannot be valid for large s unless kd/t is extremely small, in which case (3.2) becomes a still better approximation. Therefore, (3.2) should be preferred to (3.3) in essentially all computational situations.

3.2 HOMOGENEOUS LINEAR LAW PROCESS. Here are the assumptions that underlie this process.

1) All targets are identical and all searchers are identical.

2) The probability that searcher i detects target j is $\tilde{d} \in (0,1]$ for all i and j.

3) A given searcher detects different targets independently of one another.

4) A searcher making no detection is unable to attack any target.

5) If a searcher makes at least one detection then it chooses randomly, from among those targets it has detected, exactly one target to attack.

6) Given that a searcher attacks a target, the probability of destroying it is $k \in (0,1]$.

7) Searchers are mutually independent.

Assumptions 2) and 3) are in the spirit of the one-on-one detection mechanism postulated for the homogeneous linear law Markov attrition process (see page 21 above). To clarify the nature of the similarity between the two models, we offer the following comparison. For the continuous time process, the

probability that a given searcher, over a time interval of length h, detects none of t currently surviving targets is $e^{-(\tilde{r}h)t}$, where $\tilde{r}$ is the one-on-one detection rate. Therefore, the probability of at least one detection (so that a target can be attacked) is

$$(3.4) \qquad\qquad 1 - e^{-(\tilde{r}h)t} \simeq \tilde{r}ht$$

if $\tilde{r}ht$ is small. For the model defined by the preceding set of assumptions, the probability that a searcher fails to detect at least one of t targets is $(1-\tilde{d})^t$, so the probability of at least one detection (and therefore an attack) is

$$(3.5) \qquad\qquad 1 - (1-\tilde{d})^t \simeq \tilde{d}t$$

if $\tilde{d}t$ is small. The resemblance of (3.4) and (3.5) to one another and to the linear law (1.5) serves to confirm that the model at hand is a linear law model. Note also that these expressions make explicit the relation between the detection probability $\tilde{d}$ and the length h of the time period under consideration.

Note that the assumptions for the linear law process differ from those for the square law process only in the mechanism of target detection. Assumption 2) of the square law process is replaced by Assumptions 2) and 3) of the linear law process; the remaining assumptions are in one-to-one correspondence.

Although the derivation (cf.[20]) is more complicated than that of the square law model, one can show that based on the above assumptions,

$$(3.6) \qquad\qquad E[\Delta t] = t\{1-(1 - \frac{k}{t}[1-(1-\tilde{d})^t])^s\} \quad .$$

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Approximations to (3.6) include the following.

1) For large values of t or small values of $\tilde{d}$ the approximation

(3.7) $$E[\Delta t] = t\left[1-\exp\left(-\frac{ks}{t}[1-e^{-\tilde{d}t}]\right)\right]$$

holds.

2) If $\tilde{d}$ is sufficiently small that $\tilde{d}t$ and $k\tilde{d}s$ both are small, then one has the approximation

(3.8) $$E[\Delta t] \simeq k\tilde{d}ts ,$$

which, we hardly need mention, is of precisely the same form as the classical linear law (1.5).

As was true for the approximations to the analogous square law model, both (3.7) and (3.8) are better approximations as $\tilde{d}$ (i.e., the length of the time interval) decreases. However, note that (3.7) is also valid for large values of t, which is compatible with theater-level combat, whereas validity of (3.8) requires that both $\tilde{d}t$ and $k\tilde{d}s$ be small, which is not compatible with all forms of theater-level combat. In the limit as $t \to \infty$ there is also the approximation

$$E[\Delta t] \simeq ks ,$$

which is physically reasonable, but too crude even for most theater-level calculations. Moreover, an approximation of this form is valid for essentially every reasonable attrition equation, which makes it useless for comparing and understanding different equations.

The two models we have just described confirm the essential distinction between square law and linear law processes: for square law processes the rate of target detection (or acquisition or localization, to the point that an attack can be initiated) by a given searcher is not a function of the number of targets present, while for linear law processes the

rate of target detection is (at least approximately) directly proportional to the number of targets.

As we have tried to point out in Section 2, we believe that a significant advantage of dealing with attrition models in terms of sets of underlying assumption is that doing so makes generalization and extension of the models--by generalizing and extending the assumptions--natural and reasonable rather than artificial and purely formal (as happens when one simply writes down a similar-looking equation and hopes it is correct). To illustrate the point we next give a multiple shot analogue of the homogeneous linear law process.

3.3 HOMOGENEOUS LINEAR LAW PROCESS, MULTIPLE SHOT VERSION.
The assumptions are the following.

1) All targets are identical; all searchers are identical.

2) The probability that searcher i detects target j is $\tilde{d} \in (0,1]$ for all i and j.

3) A given searcher detects different targets independently of one another.

4) A searcher making no detection attacks no target.

5) If a searcher detects one or more targets, the number of those it attacks is the minimum of the number detected and a prescribed integer m. If more than m targets are detected, all subsets of size m of those detected are equally likely to be chosen to be attacked.

6) Given that it attacks a target, a searcher destroys it with probability $k \in (0,1]$.

7) Different targets attacked by a given searcher are destroyed independently.

8) Searchers are mutually independent.

The previously discussed homogeneous linear law model corresponds to the case $m = 1$. While it is easy to see on the basis of the underlying assumptions that this model indeed does generalize the preceding model to the multiple shot case in which each searcher can attack at most m of the targets it has detected, the generalization is most difficult to perceive (let alone formulate) solely on the basis of formal comparison of the equation (3.6) and the equation (3.9) below. Assumptions 1) - 4) and 6) are the same as for the single shot homogeneous linear law process; Assumption 8) here and Assumption 7) there are identical.

Subject to the assumptions above, we have

(3.9) $\quad E[\Delta t] = t[1-(1-k\tilde{d}\{B(m-1;t-1,\tilde{d}) + \frac{m}{td}B(t-m-1;t,1-\tilde{d})\})^s]$,

where

$$B(\ell;n,p) = \sum_{j=0}^{\ell} \binom{n}{j}p^j(1-p)^{n-j} ,$$

which is the cumulative distribution function for binomial probabilities with parameters n and p. For a derivation of (3.9) the reader is referred to [29]; see also [15].

The reader can verify that if $m = 1$, then (3.9) reduces to (3.6). At the other extreme, if $m = t$ (i.e., a searcher attacks every target it detects) then (3.9) becomes

(3.10) $\quad\quad\quad\quad E[\Delta t] = t[1-(1-k\tilde{d})^s]$.

The reader may have (retrospectively) observed the omission of a multiple shot square law process. This is because the square law assumptions do not in themselves define such a process, while the linear law assumptions do. For the linear law process the distribution of the number of targets detected by a particular searcher is inherently specified, but for the square law process only the probability of detecting one or more targets is specified. Specification of a distribution for the square process

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(which must be independent of the number of targets) entails additional, arbitrary assumptions that are within neither the scope nor the spirit of this paper.

Returning to the general discussion, comparison of the equations above when $d = 1$ or $\tilde{d} = 1$, as appropriate, helps further clarify the differences among them. In this case both the exact equations (3.1) and (3.6) reduce to

$$E[\Delta t] = t\left[1-\left(1-\tfrac{k}{t}\right)^s\right] ,$$

which makes it clear that square and linear law processes differ only in the mechanism of target detection. The exponential approximations (3.2) and (3.7) become

$$E[\Delta t] \simeq t\left[1 - e^{-\tfrac{ks}{t}}\right]$$

and

$$E[\Delta t] \simeq t\left[1 - e^{-\tfrac{ks}{t}(1-e^{-t})}\right] ,$$

respectively, and these two expressions are essentially equal under the conditions for their validity as approximations. On the other hand, (3.3) becomes

$$E[\Delta t] \simeq ks ,$$

while (3.8) becomes

$$E[\Delta t] \simeq kts ;$$

however, $d = 1$ or $\tilde{d} = 1$ is incompatible with conditions for validity of (3.3) or (3.8). Finally, (3.9) becomes

$$E[\Delta t] = t\left[1 - \left(1-\tfrac{km}{t}\right)^s\right] .$$

We conclude this section by discussing heterogeneous versions of square law and linear law processes.

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Let

$$M = \text{number of types of searchers,}$$
$$s(i) = \text{number of searchers of type } i,$$
$$N = \text{number of types of targets,}$$
$$t(j) = \text{initial number of targets of type } j,$$
$$\Delta t(j) = \text{attrition to targets of type } j.$$

Also, let

$$t = \sum_{j=1}^{N} t(j)$$

be the total initial number of targets. Recall that only the attritions $\Delta t(j)$ are random variables.

3.4 HETEROGENEOUS SQUARE LAW PROCESS.

The assumptions in this case are essentially the same as those of the homogeneous square law process described earlier in this section. For searchers of type i, the probability of detecting one or more targets is $d_i \in (0,1]$ and does not depend on the numbers of targets present. Given that a searcher of type i attacks a target of type j, the target is destroyed with probability k_{ji}. We emphasize that the assumption that all searchers be mutually probabilistically independent remains in force. It then follows that for each j

$$(3.11) \qquad E[\Delta t(j)] = t(j) \left[1 - \prod_{i=1}^{M} \left(1 - \frac{k_{ji} d_i}{t} \right)^{s(i)} \right];$$

the reader is referred to [24] for a derivation.

If each of the quantities $k_{ji} d_i / t$ is small (which certainly occurs in theater-level calculations over relatively short time intervals) then (3.11) admits the exponential approximation

$$(3.12) \quad E[\Delta t(j)] \approx t(j) \left[1 - \exp\left(-\frac{1}{t} \sum_{i=1}^{M} k_{ji} d_i s(i) \right) \right],$$

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which will figure in the discussion in Section 4 of the attrition equations used in several theater-level combat models. Under the condition that $k_{ji}d_i s(i)/t$ be small for each i, one has the approximation

$$(3.13) \qquad E[\Delta t(j)] \simeq \frac{t(j)}{t} \sum_{i=1}^{M} k_{ji}d_i s(i)$$

to (3.11). It should not be concluded from the appearance of t(j) as a factor on the right-hand side of (3.13) that this equation corresponds to a linear law process. In fact, the underlying assumptions clearly imply that the factor $t(j)/t$ represents a proportional allocation of fire of the same sort that appears in (2.14) and (2.15).

However, (as discussed further in Sections 2 and 4) alternative allocations of fire are desirable and possible. Assume that, given that a searcher of type i detects one or more targets, it chooses to attack a target of type j with probability α_{ji}. Then equation (3.11) becomes

$$(3.11') \qquad E[\Delta t(j)] = t(j)\left[1 - \prod_{i=1}^{M}\left(1 - \frac{k_{ji}\alpha_{ji}d_i}{t(j)}\right)^{s(i)}\right].$$

The approximations (3.12) and (3.13) become, respectively,

$$(3.12') \quad E[\Delta t(j)] \simeq t(j)\left[1 - \exp\left(-\frac{1}{t(j)}\sum_{i=1}^{M} k_{ji}\alpha_{ji}d_i s(i)\right)\right]$$

and

$$(3.13') \qquad E[\Delta t(j)] \simeq \sum_{i=1}^{M} k_{ji}\alpha_{ji}d_i s(i).$$

For the proportional allocation of fire given by $\alpha_{ji} = t(j)/t$ for all i and j, (3.11) - (3.13) are recovered. A general family of allocations is discussed in Section 4.

3.5 HETEROGENEOUS LINEAR LAW PROCESS. The assumptions underlying this process are those underlying the homogeneous linear law process (page 36 above) with modifications to permit heterogeneous forces. Each searcher can still attack at most one target and the probability that a particular searcher of type i detects a particular target of type j is $\tilde{d}_i \in (0,1]$. We emphasize that this probability depends only on the type of searcher. Target dependent detection probabilities in general render the process intractable (cf. [20]), although it is possible to handle the case where the searcher i-target j detection probability is either $\tilde{d}_i$ or zero. See [20] for some other variations on the hypotheses. The probability that a searcher of type i destroys a target of type j, given an attack, is k_{ji} (it is essentially always possible to allow kill probabilities to depend on the type of target). These assumptions imply that for each j

$$(3.14) \quad E[\Delta t(j)] = t(j)\left[1 - \prod_{i=1}^{M}\left(1 - \frac{k_{ji}}{t}[1-(1-\tilde{d}_i)^t]\right)^{s(i)}\right];$$

a derivation may be found in [20]. The reader will note the close resemblance between (3.14) and (3.6); of course, if M = 1, the former reduces to the latter.

Two exponential approximations to (3.14) are in use. The more commonly used is given by

$$(3.15) \quad E[\Delta t(j)] \approx t(j)\left[1-\exp\left(-\sum_{i=1}^{M} k_{ji}\tilde{d}_i s(i)\right)\right].$$

Conditions for validity of (3.15) are that $\tilde{d}_i t$ be small for each i and that $\tilde{d}_i k_{ji}$ be small for each i and j. Clearly the first of these (which requires that $\tilde{d}_i$ be extremely small) essentially always implies the second. The less frequently used approximation is valid under wider conditions; it requires

only that each $\tilde{d}_i$ be small (and improves as t increases as well) and is given by

$$(3.16) \quad E[\Delta t(j)] \simeq t(j)\left[1-\exp\left\{-\frac{1}{t}\sum_{i=1}^{M} k_{ji}s(i)(1-e^{-\tilde{d}_i t})\right\}\right].$$

Finally, if $\tilde{d}_i t$ is small for all i and $k_{ji}\tilde{d}_i s(i)$ is small for each i and j, then (3.14) may be approximated using the equation

$$(3.17) \qquad E[\Delta t(j)] \simeq t(j)\sum_{i=1}^{M} k_{ji}\tilde{d}_i s(i) .$$

While this expression is definitely of linear law form (compare with (1.15)) the conditions for its validity will often not obtain in models of theater-level combat. Of the three approximations, (3.16) is definitely the best in theater-level contexts, with (3.15) next best and (3.17) worst.

Development of a multiple shot analogue of the heterogeneous linear law process should be relatively straightforward, but we have not worked out the details. The equation would be somewhat more complicated than (3.9). (A corresponding analogue of the heterogeneous square law process involves the same conceptual and practical difficulties that were mentioned earlier for the homogeneous case.)

Allocations of fire similar to those appearing in (3.11') - (3.13') could, but only after further research, be incorporated into the heterogeneous linear law model. Care must be exercised because the linear law target detection mechanism already implies a particular joint distribution of the numbers of targets detected by a particular searcher and consistency with it must be maintained. The probability that a target of type j is chosen to be attacked must be allowed to depend on the numbers of targets (of all types) detected.

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To conclude this section we observe for one final time that the <u>only</u> difference between corresponding square law and linear law processes is the mechanism of target detection. For square law processes the probability that a searcher detects one or more targets (i.e., the probability that it is able to <u>engage</u> some target) is independent of the numbers of targets, while for linear law processes this engagement probability is of the form $1-(1-\tilde{d})^t$. In the same way that (3.1) and (3.6) coincide when $d=\tilde{d}=1$, (3.11) with all $d_i=1$ is identical to (3.14) with all $\tilde{d}_i=1$. The practical consequence is that for modeling combat processes in which detection is virtually certain, square and linear law processes do not differ significantly.

The simplicity of the attrition equations described in this section permits their use as the basis for calculating attrition in highly complex models of theater-level combat, as we describe in the next section.

4.   ATTRITION COMPUTATIONS IN THEATER-LEVEL COMBAT MODELS

In this section we discuss the attrition calculations in
four principal computerized simulations of ground-air combat
at the theater level, namely the CONAF Evaluation Model (CEM IV),
the IDAGAM I Model, the Lulejian-I Model, and the VECTOR-2 Model.
We shall show that the simplified stochastic attritions models
described in Section 3 above constitute the core of the calcula-
tions in the first 3 of these models (i.e., all except the
VECTOR-2 Model) and that the fourth uses, in addition, attrition
calculations based on the stochastic attrition processes discussed
in Section 2.   Therefore, the sets of assumptions given in the
preceding sections serve to explain the attrition calculations
in four principal theater-level combat models in current use.

By theater-level combat we mean bilateral combat between
air and ground forces (with multiple types of weapons) over a
large geographical region over a period of several days.   A
model of theater-level combat must account for force movements
and attrition in both forward and rear areas; typically (pos-
sibly rudimentary) representations of logistics and tactics
are also present.   The complexity of the force accounting,
force movement and decision-making structures in a theater-
level model render mandatory a simplified representation of
attrition effects in order to satisfy running time and storage
constraints on the model.   In practice this means that sophis-
ticated depictions of attrition such as Monte Carlo simulations
must be foregone and that attrition must be calculated with
expected-value attrition equations that are relatively simple.
Consequently, the equations discussed in Section 3 are par-
ticularly suited to modeling of theater-level combat.

And indeed, as we discuss below, the equations of Sections
2 and 3 are the basis of the attrition calculations in the four
models listed above.   We emphasize, however, that the descrip-
tions given in this section are highly abstracted and that we,

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in particular, omit discussion of the explicit form of the dependence of effectiveness parameters on weapon type and the combat situation, or additional dependence on variables such as terrain and posture. Our goal is to indicate the form of the equations used to perform the calculations and to demonstrate how that form is related to the material developed above; hence many important modeling issues are ignored entirely. For more detailed descriptions of the attrition calculations, the reader is referred to the model documentation published by the developers ([32], [1,2,5], [33], [34,35]), to the author's critiques of the models ([25], [28], [24], [30]), on which this section is largely based, and to the "Four Model Comparison Study" reported in [46]. A recent report treating the same models is the GAO report [10].

All four models are iterative, fixed time step computerized simulations of ground-air combat at the theater level. Each breaks down the combat into (somewhat) smaller-sized interactions in which attrition effects are computed. Furthermore, attrition is generally calculated by applying a unilateral attrition equation first to one side and then (with searchers and targets interchanged and parameter values possibly changed) to the other side. In both calculations the initial numbers of combatants are used as inputs. To illustrate, if for example the homogeneous square law equation (3.2) were used to represent combat between r Red combatants and b Blue combatants, then the resulting attritions would be

$$(4.1a) \qquad E[\Delta b] = b\left[1 - \left(1 - \frac{k_1 d_1}{b}\right)^r\right]$$

and

$$(4.1b) \qquad E[\Delta r] = r\left[1 - \left(1 - \frac{k_2 d_2}{r}\right)^b\right],$$

respectively, where d_1, d_2 are detection probabilities and k_1, k_2 are conditional probabilities of kill given attack.

It suffices, therefore, to describe for each model relevant unilateral attrition equations in terms of targets and searchers (i.e., in the language of Section 3). In passing, we note that none of the models uses asymmetric attrition structures, even though there are interactions, such as that in air-to-air combat between interceptors and escorts, with evident physical and combat objective asymmetries. One way of dealing with these kinds of interactions is described in [3].

Following the notation of Section 3 let

M = number of types of searchers,

$s(i)$ = number of searchers of type i,

N = number of types of targets,

$t(j)$ = initial number of targets of type j.

Let

$$t = \sum_{j=1}^{N} t(j)$$

be the total initial number of targets and, finally, let

$\delta t(j)$ = calculated attrition to targets of type j.

We have introduced the new notation $\delta t(j)$ and terminology "calculated attrition" to emphasize that most of the attrition equations discussed below are inexact, either because they are approximations to begin with or because of difficulties arising from iterative use of them. While we will not discuss this latter problem further, it may be significant and is certainly worthy of some research effort.

Finally, we point out that our notation matches that of Section 3, not those of the model documentations.

We now proceed to describe the attrition calculations in the four models.

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4.1 THE CONAF EVALUATION MODEL. *For the CONAF Evaluation Model (CEM), the situation is simple to describe. Essentially all attrition calculations in this model are (cf. [32],[25]) performed using equations of the form

$$(4.2) \qquad \delta t(j) = t(j) \left[1 - \exp\left(-\frac{\alpha_j}{t} \sum_{i=1}^{M} p_{ji} s(i)\right) \right] ,$$

where p_{ji} is the "potential of one searcher of type i against targets of type j" and is computed within the model from more primitive and physically oriented parameters, and where α_j is a damage coefficient. The equation (4.2) is the exponential approximation to the heterogeneous square law process of Section 3 that appears as equation (3.12). Observe that there is no direct correspondence between the parameters in (3.12) and those in (4.2). Moreover, the assumptions leading to (3.12) permit one parameter (representing detection or acquisition) depending on searcher type alone and one parameter (representing destruction given an attack) depending on the types of both the searcher and the target, whereas (4.2) contains one parameter depending on both types and one depending only on the type of the target. This raises the possibility of a conceptual error by the developers of the CEM, which might have occurred because there was no articulation or understanding of the physical assumptions underlying the equation to which (4.2) is an approximation.

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*The author has been informed by P.E. Louer (personal communication) that as of early 1981 a new attrition structure for the CEM is under development.

4.2 THE IDAGAM I MODEL.

We will first discuss the equations used to describe air-to-air combat, and then those used to represent ground combat. Important references are [1], [2], [5], and [28]. The IDAGAM I and IDAGAM II models do not differ in their attrition structures, so the comments below are applicable to both.

AIR COMBAT. In the IDAGAM I model there is available a number of attrition equations for calculations concerning air-to-air combat. Although the IDAGAM documentation is not as clear as one would hope in explaining the differences and relationships among the various alternatives, it does contain some guidance. The first available choice, and the equation used in most implementations of the model, is

$$(4.3) \qquad \delta t(j) = t(j) \left[1 - \prod_{i=1}^{M} \left(1 - \frac{k_{ji}}{t} [1 - (1-\tilde{d})^t] \right)^{s(i)} \right],$$

where $\tilde{d}$ is a detection probability depending on neither the type of the searcher nor the type of the target. Equation (4.3) is an exact equation and is the form to which the heterogeneous linear law equation (3.14) reduces when the detection probability does not depend on the type of the searcher.

Partly for the sake of comparisons but also, in the opinion of the author, somewhat unnecessarily and possibly even misleadingly, the IDAGAM I model contains two approximations to (4.3), namely the equations

$$(4.4) \qquad \delta t(j) = t(j) \left[1 - \exp\left(-\frac{1}{t} \sum_{i=1}^{M} k_{ji} s(i) \left(1 - e^{-\tilde{d}t} \right) \right) \right]$$

and

$$(4.5) \qquad \delta t(j) = t(j) \tilde{d} \sum_{i=1}^{M} k_{ji} s(i) .$$

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These equations were discussed in Section 3, where they are numbered as (3.16) and (3.17), respectively. As pointed out in Section 3, the two approximations are not equally good; while (4.4) is valid (provided the time interval is short) for parameter values and numbers of searchers and targets likely to be present in theater-level contexts, (4.5) is not.

There is a second attrition fundamentally different from (4.3) that is also available in IDAGAM I as an option for use in calculating air-to-air attrition, and that equation is

$$(4.6) \qquad \delta t(j) = t(j) \left[1 - \exp\left(-\frac{1}{t} \sum_{i=1}^{M} k_{ji} d_i s(i) \right) \right] ,$$

which is the exponential approximation (3.12) to the heterogeneous square law model (3.11) and is nearly the equation used in the CEM. However, the parameter dependences in (4.6), in which the d_i are detection probabilities and the k_{ji} probabilities of target destruction given attack, correctly match those in (3.12). Unfortunately, the exact square law equation (3.11) is not available in IDAGAM I, even though the linearized approximation (which is less widely applicable than (4.6))

$$(4.7) \qquad \delta t(j) = \frac{t(j)}{t} \sum_{i=1}^{M} k_{ji} d_i s(i)$$

is available. Of course, (4.7) is precisely the same as (3.13).

If, on the basis of underlying assumptions, one believes that the heterogeneous square law model embodied in (3.11) is an appropriate description of air-to-air combat, then in using IDAGAM I one cannot choose the exact equation, but is forced instead to choose between the two approximations (4.6) and (4.7), which are not (as was discussed in Section 3) equally accurate. At least, however, the available equations are approximations to an exact equation resulting from a known set of assumptions.

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Finally, there is also available the equation

$$(4.8) \qquad \delta t(j) = t(j) \left[1 - \prod_{i=1}^{M} (1-d_i k_{ji})^{s(i)} \right] ,$$

which is an exact heterogeneous version of the Linear Law Process, Multiple Shot Version, discussed in Section 3, with m = t, so that every searcher can attack each target it detects. The equation most analogous to (4.8) that appears explicitly in Section 3 is (3.10).

GROUND COMBAT. Ground combat attrition can be calculated in IDAGAM I in two essentially different ways, the first of which will be mentioned only briefly. In the first method potential weapon systems losses and actual casualties (to either personnel or aggregated firepower) are calculated separately (the latter from force ratios and historical data); actual casualties are then used to scale potential weapons systems losses in order to obtain actual weapons system losses. In this method, the equation used to calculate potential weapons systems losses is of the same form as the equation discussed in the next paragraph.

The second method for calculating ground combat attrition uses equations of the form

$$(4.9) \qquad \delta t(j) = \sum_{i=1}^{M} \alpha(j,i) k_{ji} s(i) ,$$

where the k_{ji} are probabilities of kill and the $\alpha(j,i)$ are fire allocation factors of the following form:

$$(4.10) \qquad \alpha(j,i) = \frac{t(j) \beta^*(j,i)/t^*(j)}{\displaystyle\sum_{k=1}^{N} t(k) \beta^*(k,i)/t^*(k)} .$$

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In this expression, $\alpha(j,i)$ is the fraction of searchers of type i that actually fire upon targets of type j and $\beta^*(j,i)$ is the corresponding fraction when the force of targets is a prescribed "standard" force $(t^*(1), \ldots, t^*(N))$. The reader is referred to [19] for a proof that all fire allocations satisfying a small set of simple, intuitive axioms are of the form (4.10). By way of illustration, if

$$\beta^*(j,i) = \frac{t^*(j)}{\sum\limits_{k=1}^{N} t^*(k)} \,,$$

then (4.10) yields

$$\alpha(j,i) = \frac{t(j)}{\sum\limits_{k=1}^{N} t(k)} \,.$$

In other words, if the "standard" allocation is proportional, so is the "actual" allocation.

We note in passing that allocations of the form (4.10) could replace the proportional allocations in equations (2.14) and (2.15).

Returning to the main exposition, the equation (4.9) is of the form of the approximation (3.13') to the exact heterogeneous square law equation (3.11'). The discussion in Section 3 suggests that a corresponding exponential approximation might be preferable.

4.3 THE LULEJIAN-I MODEL.

Attrition calculations in the Lulejian-I model ([33],[24]) are virtually all performed using equations of the form

$$(4.11) \qquad \delta t(j) = t(j) \left[1 - \exp\left(-\frac{1}{\rho t} \sum_{i=1}^{M} \bar{k}_i s(i) \right) \right] ,$$

where ρ is a separation distance and where

$$(4.12) \qquad \bar{k}_i = \frac{1}{t} \sum_{\ell=1}^{N} k_{\ell i}^* t(\ell) .$$

The quantity $k_{\ell i}^*$ is the "potential of one searcher of type i to destroy targets of type ℓ," so one should interpret $\bar{k}_i$ as the average destruction potential of one searcher of type i, weighted by the relative numbers of targets present.

The equation (4.11) is the exponential approximation (3.12) to the exact heterogeneous square law equation (3.11), but includes the unnecessary, and in fact incorrect, parameter averaging given by (4.12).

In the three models considered so far, the dominant attrition equation is some form, usually an approximation, of the heterogeneous square law model (3.11); only IDAGAM I provides in some cases the linear law alternative (3.14). Understanding the assumptions leading to (3.11) and (3.14) provides one with essentially complete knowledge of the basis for attrition calculations in the CEM, IDAGAM I and Lulejian-I models.

4.4 THE VECTOR-2 MODEL.

Here the equations for ground combat are treated first.

GROUND COMBAT. Attrition calculations for ground combat in the VECTOR-2 model ([34],[35]) are in several ways more sophisticated and complicated than those in the other three models, although not without some unique difficulties as well.

Not only can the time scale be much shorter, which is certain to reduce errors resulting from the very nature of the iterative calculations in all four models, but also there is explicit representation of physical effects such as parallel or serial target acquisition, alternating periods of target visibility and invisibility, random times to acquire targets given continuous visibility, random times to destroy a target given continuous acquisition, and possible destruction of a target by another searcher. In addition to positive consequences, this complexity makes it uncertain whether there exists a definitive set of underlying assumptions. Indeed, since the equations are used to represent manifestly transient phenomena, but are based partly on asymptotic properties of certain stochastic processes, there is some reason to doubt that a set of underlying assumptions exists.

In this paper we cannot discuss in detail either the advantages or the disadvantages of the method of attrition calculation used in VECTOR-2, but we will at least describe the flavor of it. For details the reader is referred to [30], [34], and [35].

Ground combat attrition equations in the VECTOR-2 model are of the general form

$$(4.13) \qquad \delta t(j) = \sum_{i=1}^{M} a(j,i,\underline{t},\underline{s})s(i) \, ,$$

where a is an <u>attrition rate function</u>,

$$\underline{t} = (t(1), \ldots, t(N))$$

is the entire target force (type-by-type), and

$$\underline{s} = (s(1), \ldots, s(M))$$

is the entire force of searchers. It is tempting, but not justified, to conclude that (4.13) is essentially the deterministic square law model given by (1.10). To do so would be

erroneous first of all because the attrition calculations of VECTOR-2 are asserted in [34] to be fundamentally based on stochastic considerations. Although one can dispute this claim, it does seem reasonable to view (4.13) as a modification of the heterogeneous square law process with generator (2.13). The more fundamental reason why it is incorrect to view (4.13) as merely a square law equation is that the attrition rate functions $a(j,i,\underset{\sim}{t},\underset{\sim}{s})$ depend nontrivially on the target force $\underset{\sim}{t}$ and the searcher force $\underset{\sim}{s}$. Only if there were dependence on neither would (4.13) be precisely a square law equation.

To give an idea of the nature of the attrition rate functions used in the VECTOR-2 model, let us consider the case of searchers of type i that use parallel target acquisition. This means that target types are ranked in order of priority and that when a searcher engages a target it continues to search for targets of higher priority. Should it acquire one it immediately switches its fire to the newly acquired target. The target priorities must be prescribed by the model user and represent nothing physical; the idea that a weapons system might cease engaging a nearby and threatening target because a higher priority target appears in the distance is distinctly unappealing, but that is how the parallel target acquisition scheme operates in VECTOR-2. One then has

$$(4.14) \qquad a(j,i,\underset{\sim}{t},\underset{\sim}{s}) = \alpha_{ji}\left[1 - \left(1 - \frac{n_{ji}}{\mu_{ji}+n_{ji}}\frac{\lambda_{ji}}{\mu_{ji}+\lambda_{ji}}\right)^{t(j)}\right]$$

$$\times \left[\Pi^*\left(1 - \frac{n_{ki}}{\mu_{ki}+n_{ki}}\frac{\lambda_{ki}}{\mu_{ki}+\lambda_{ki}}\right)^{t(k)}\right],$$

where the product Π^* is over all k for which targets of type k are higher priority than those of type j, and where

$\bar{u}_{ji}$ = mean length of visible periods for one target of type j and one searcher of type i,

$\bar{n}_{ji}$ = mean length of invisible periods for one target of type j and one searcher of type i,

$\bar{\lambda}_{ji}$ = mean time required for one searcher of type i to acquire one target of type j, given continuous visibility,

$\bar{\alpha}_{ji}$ = mean time required for one searcher of type i to destroy one target of type j, given continuous acquisition.

The three factors on the right-hand side of (4.14) may be interpreted as the kill rate, the probability that a type j target is visible and acquired (and is hence under attack), and the probability that no higher priority target is visible and acquired. We refer the reader to [6], [30], and [34] for details of the interpretation. Note, however, the following anomaly: as $\alpha_{ji} \to \infty$, $a(j,i,\underline{t},\underline{s}) \to \infty$, which is not sensible. As $\alpha_{ji} \to \infty$, target destruction consumes a negligible amount of time, but target destruction should not occur at an infinite rate since it would still be limited by both invisibility and loss of acquisition.

For descriptions of attrition rate functions for searchers with serial target acquisition we refer the reader to [30], [34], and [35].

AIR COMBAT. Attrition in air-to-air combat is calculated in the VECTOR-2 model using equations of the form

$$(4.15) \qquad \delta t(j) = \frac{t(j)}{t} \sum_{i=1}^{M} c(j,i) \min \{t(j), m_i\} s(i) ,$$

where the $c(j,i)$ are positive constants, interpreted in [34] as "attrition caused by one [searcher] of type i that engages only [targets] of type j," and where m_i is the maximum number of targets that a searcher of type i can engage simultaneously.

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The equation (4.15) is a linearized approximation to one heterogeneous version (there are several heterogeneous versions that make sense) of the "Homogeneous Linear Law Process, Multiple Shot Version" described in Section 3. To fully justify this assertion would require an unreasonable amount of space and additional equations, which is not merited at this point in the paper. The unconvinced reader can find some details in [34] and [35]. Validity of the approximation (4.15) is questionable in the theater-level context, even for the relatively smaller pieces into which engagements are decomposed in VECTOR-2, and lack of an exact equation and a set of underlying assumptions are significant shortcomings of the model.

To conclude this section, we observe once again that the simplified attrition models treated in Section 3 include nearly all the equations used in four principal theater-level simulations of ground-air combat. The assumptions set forth in Section 3 contribute much to understanding differences and similarities among the attrition calculations in these four models.

5. CONCLUSION

In this paper we have surveyed several aspects of the theory and applications of Lanchester models of combat, beginning with the original deterministic systems of differential equations, continuing through continuous time Markov process analogues, proceeding to simplified stochastic models, and concluding with a discussion demonstrating that virtually all attrition calculations in four principal ground-air combat models are performed using one of the simplified stochastic models or an approximation to one of the simplified stochastic models. The approach and philosophical basis of this paper are distinctive in seeking to illuminate the nature of the various attrition models in terms of sets of underlying assumptions. Analysis and understanding of assumptions can only lead to more enlightened use of models.

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